THEORY OF THE ELASTIC MODULI OF A HETEROGENEOUS MEDIUM

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It is shown that the theory of random functions permits the expansion of the effective tensor λ^*_{ijkl} for the elastic moduli with respect to correlation functions and that it leads in the second approximation in the Voigt-Reuss scheme to values that lie to one side of the λ^*_{ijkl}, while in the third approximation it brackets the latter. The analysis is used to refine the Hashin limits to the elastic moduli for a mechanical mixture of isotropic components and polycrystalline aggregates of cubic structure.

There are two methods for calculating the effective elastic moduli of heterogeneous solids: virial expansion [1] (as a power series in the concentration of one of the components) and the method of correlation functions [2] (expansion with respect to relative fluctuation of the elastic moduli). Identical results should be obtained in the two cases if all terms are incorporated, but great mathematical difficulties restrict one to the lowest approximations. The first approximation in the virial method gives better results when the concentration of one component is low, while the method of correlation functions gives better results when the fluctuations in the elastic moduli are small and the concentrations are similar.

Methods have been developed for determining the upper and lower bounds in both approaches, and various schemes of averaging are used for this purpose in the correlation-function method. The upper bound is established by renormalizing the equation of equilibrium, while the lower one is found by renormalizing the equation of incompatibility. The range of the bracketing can be reduced by means of higher approximations. The range can be reduced in the limit to zero, which implies passing from an approximate effective tensor to the true one, which relates the means in stress and strain over the material. Here we show that the two methods of renormalization give identical results when all terms of the series are summed.

If the tensor has a Gaussian distribution, the moment functions of odd order are zero, while the even ones are expressed via combinations of the binary functions [3]. However, a mechanical mixture of several components is not Gaussian, and the odd moments are not zero. Splitting of the higher-order correlation functions is possible also for mechanical mixtures having determinate phase interfaces, but this involves various simplifying assumptions. A derivation is given for a moment of arbitrary order, which allows one to formulate the conditions under which such splitting is possible. The results are used in calculating the exact value of the effective bulk modulus for a medium with a homogeneous shear modulus.

1. To calculate higher-order correlation functions we seek the probability that at point $\mathbf{r}_1$ there is component i, at point $\mathbf{r}_2$ component j, etc. The joint probability of this is denoted by $P_{ijk\ldots}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\ldots)$. We derive the differential equations satisfied by $P_i(\mathbf{r}_1)$, $P_{ij}(\mathbf{r}_1, \mathbf{r}_2)$, etc. via the conditional probability $p_{ij}(\mathbf{r}_1|\mathbf{r}_2)$, which defines the probability of finding component i at $\mathbf{r}_1$ if component j is at $\mathbf{r}_2$. We consider only a two-component system, so i, j = 1, 2. Then the probability $P_i(\mathbf{r} + d\mathbf{r})$ of finding component i at point $\mathbf{r} + d\mathbf{r}$ is expressed in terms of the probability $P_j(\mathbf{r})$ of finding component j at $\mathbf{r}$ by Smoluchowski's equation:

$$P_i(\mathbf{r} + d\mathbf{r}) = \Sigma p_{ij}(\mathbf{r} + d\mathbf{r}\,|\,\mathbf{r}) P_j(\mathbf{r}). \qquad (1.1)$$

We assume that the mixture obeys the condition

$$p_{ij}(\mathbf{r} + d\mathbf{r}\,|\,\mathbf{r}) = \delta_{ij} + \mathbf{A}_{ij}(\mathbf{r})\,d\mathbf{r}, \qquad (1.2)$$

in which the matrix $\mathbf{A}_{ij}(\mathbf{r})$ is defined by

$$\mathbf{A}_{ij}(\mathbf{r}) = \left[\frac{\partial}{\partial \rho} p_{ij}(\rho\,|\,\mathbf{r})\right]_{\rho=\mathbf{r}} \quad \left(\sum_i \mathbf{A}_{ij} = 0\right) \qquad (1.3)$$

and satisfies the normalization condition stated in parentheses.

For an isotropic and quasi-homogeneous medium

$$\mathbf{A}_{ij}(\mathbf{r}) = A_{ij}(0)\frac{\mathbf{r}}{r} = A_{ij}\,\mathbf{n}. \qquad (1.4)$$

We substitute (1.2) into (1.1) to get

$$\nabla P_i(\mathbf{r}) = \sum_j \mathbf{A}_{ij} P_j(\mathbf{r}). \qquad (1.5)$$

We solve (1.5) with (1.4) and the normalization conditions to get

$$P_1 = aA_{12} + C_1 \exp(-r/a),$$
$$P_2 = aA_{21} + C_2 \exp(-r/a),$$
$$a \equiv (A_{12} + A_{21})^{-1}. \qquad (1.6)$$

Then, in view of the condition of quasi homogeneity,

$$P_i(\mathbf{r}) = c_i, \qquad (1.7)$$

we find the relation of A_{ij} to the component concentrations

$$aA_{ij} = \begin{Vmatrix} -c_2 & c_1 \\ c_2 & -c_1 \end{Vmatrix}. \qquad (1.8)$$

We use the following equation to calculate the joint probability $P_{ij}(\mathbf{r}_1, \mathbf{r}_2)$:

$$P_{ij}(\mathbf{r}_1, \mathbf{r}_2) = \sum_k p_{ik}(\mathbf{r}_1\,|\,\mathbf{r}_3) P_{kj}(\mathbf{r}_3, \mathbf{r}_2). \qquad (1.9)$$

Then, as $P_{kj}(\mathbf{r}_1, \mathbf{r}_2) = P_{kj}(|\mathbf{r}_1 - \mathbf{r}_2|)$ for a quasi-homogeneous medium, we have

$$\frac{d}{dr} P_{ij}(r) = \sum_k A_{ik} P_{kj}(r). \qquad (1.10)$$

Substitution for A_{ik} from (1.8) gives

$$a\frac{d}{dr} P_{ij}(r) = -P_{ij}(r) + P_i P_j. \qquad (1.11)$$

The following is the solution to (1.11) that satisfies $P_{ij}(0) = c_i\delta_{ij}$:

$$P_{ij}(r) = \alpha_{ij} + \beta_{ij}\varphi(r), \tag{1.12}$$

$$\varphi(r) \equiv \exp(-r/a),$$

$$\alpha_{ij} = c_i c_j, \quad \beta_{ij} = c_i\delta_{ij} - c_i c_j. \tag{1.13}$$

Then (1.12) gives the joint probability found under the condition that the passage from the initial state to the final state is performed by motion of a single point by **r**. Now let the transition be performed by change in the coordinates of the start and end of the line, i. e., there is a transition from the state $\mathbf{r}_3$, k and $\mathbf{r}_4$, l to $\mathbf{r}_1$, i and $\mathbf{r}_2$, j. This approach allows us to introduce the four-subscript matrix A_{ijkl}, which will be used in what follows to calculate the joint probability $P_{ijk}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$. Then (1.9) is replaced by

$$P_{ij}(\mathbf{r}_1, \mathbf{r}_2) = \Sigma p_{ijkl}(\mathbf{r}_1, \mathbf{r}_2 \mid \mathbf{r}_3, \mathbf{r}_4) P_{kl}(\mathbf{r}_3, \mathbf{r}_4). \tag{1.14}$$

To calculate the matrix for transition from state kl, **r** to the new state ij, **r** + d**r** we resolve d**r** into the normal component $dr_n = d\mathbf{r} \cdot \mathbf{n}$ and the tangential component $dr_\tau = r d\mathbf{n}$. If $dr_\tau = 0$, the transition occurs between points lying on the ray $\mathbf{r}/r$ = const, while $dr_n = 0$ means transition between points on the sphere r = const. The first transition alters the ratio of the radius to the scale of the correlations, while the second leaves it unchanged.

We therefore consider transitions as identical if the initial and final states have the same ratio of the distances to the scale of the correlations. A sphere represents the surface formed by the correlation-scale vector for an untextured medium, so we can put $dr_\tau = 0$ and consider only transitions that alter the absolute magnitude of the distance between points.

The following matrix then describes the transition from kl, **r** to ij, **r** + d**r**:

$$p_{ijkl}(\mathbf{r} + d\mathbf{r} \mid \mathbf{r}) = \delta_{ik}\,\delta_{jl} + A_{ijkl}\,dr, \tag{1.15}$$

and substitution into (2.14) gives

$$\frac{d}{dr} P_{ij} = \sum_{k,l} A_{ijkl} P_{kl}. \tag{1.16}$$

The equivalence of (1.16) and (1.10) means that A_{ijkl} takes the form

$$a A_{ijkl} = -(\delta_{ik} - P_i)(\delta_{jl} - P_j). \tag{1.17}$$

Relation (1.15) imposes severe restrictions on the structure of the mixture. If we take account only of the mean grain sizes of the components in calculating the higher moments (these sizes are related to the correlation scale by the rule for mechanical mixing), this is equivalent to ignoring the grain shape, which can be taken into account, for example, via the set of moments $\langle R(\omega)\rangle$, $\langle R(\omega_1)R(\omega_2)\rangle$ etc., where R is the coordinate of the surface of a grain in a frame of reference whose origin coincides with the center of gravity, ω = R/R, and $\langle\ \rangle$ denotes averaging with respect to angle. Moreover, no distinction is drawn between the matrix and an inclusion, so the coupling is considered identical for the regions filled by the two components. Equation (1.16) is applicable if the component concentrations do not differ too greatly and the components have similar degrees of coupling. The method should give the best results for quasi-spherical grains, when $\langle (R - \langle R\rangle)^2\rangle \ll \langle R\rangle^2$.

2. We now consider the calculations of $P_{ijk}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$. As the medium is isotropic and quasi homogeneous, $P_{ijk}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ can be put as $P_{ijk}(r', r'', r''')$, where

$$r' = |\mathbf{r}_2 - \mathbf{r}_3|, \quad r'' = |\mathbf{r}_3 - \mathbf{r}_1|, \quad r''' = |\mathbf{r}_1 - \mathbf{r}_2|.$$

We envisage transitions such that r″ = const and r‴ = const but r′ ≠ const. Arguments similar to those for (1.15) and (1.17) then give

$$\frac{\partial}{\partial r'} P_{ijk}(r', r'', r''') = \sum_{m,n} A_{jkmn} P_{imn}(r', r'', r'''). \tag{2.1}$$

We substitute from (1.17) for the A_{jkmn} to get

$$\frac{\partial}{\partial r'} P_{ijk}(r', r'', r''') = -\frac{1}{a}\{P_{ijk}(r', r'', r''') -$$

$$- P_j P_{ik}(r'') - P_k P_{ij}(r''') + P_i P_j P_k\}. \tag{2.2}$$

This allows us to calculate the third-order correlation function for the tensor. We introduce moment N of the tensor in order to find the differential equation for the explicit form of this function. For simplicity we omit the tensor subscripts $\lambda^{i}_{pqrs} \equiv \lambda_i$ and get for this moment that

$$M_{\alpha_1\alpha_2\ldots\alpha_n}(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \langle \lambda^{\alpha_1}(\mathbf{r}_1)\lambda^{\alpha_2}(\mathbf{r}_2)\ldots\lambda^{\alpha_n}(\mathbf{r}_n)\rangle =$$

$$= \lambda_i^{\alpha_1}\lambda_j^{\alpha_2}\ldots\lambda_q^{\alpha_n} P_{ij\ldots q}(\mathbf{r}_1, \ldots \mathbf{r}_n) \tag{2.3}$$

with $\sum_{k=1}^{n} \alpha_k = N$. The centered moment of order N is then given by

$$\mu_{\alpha_1\alpha_2\ldots\alpha_n}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_n) =$$

$$= \langle [\lambda(\mathbf{r}_1) - \langle\lambda(\mathbf{r}_1)\rangle]^{\alpha_1} [\lambda(\mathbf{r}_2) -$$

$$- \langle\lambda(\mathbf{r}_2)\rangle]^{\alpha_2} \ldots [\lambda(\mathbf{r}_n) - \langle\lambda(\mathbf{r}_n)\rangle]^{\alpha_n}\rangle =$$

$$= \sum_{i,j,\ldots,q} [\lambda_i - \langle\lambda(\mathbf{r}_1)\rangle]^{\alpha_1} \ldots [\lambda_q -$$

$$- \langle\lambda(\mathbf{r}_n)\rangle]^{\alpha_n} P_{ij\ldots q}(\mathbf{r}_1, \ldots, \mathbf{r}_n). \tag{2.4}$$

We get from (2.3) and (2.4) for an isotropic quasi-homogeneous medium that

$$M_N(\mathbf{r}) = <\lambda^N(\mathbf{r})> = \sum \lambda_i^N P_i(\mathbf{r}) = \sum \lambda_i^N c_i,$$

$$\mu_1 = 0, \quad \mu_2 = M_2 - M^2{}_1,$$

$$\mu_3 = M_3 - 3M_1M_2 + 2M^3{}_1,$$

$$M_{11}(\mathbf{r}_1, \mathbf{r}_2) = M^2{}_1 + \mu_2 \exp(-r'''/a),$$

$$\mu_{11}(\mathbf{r}_1, \mathbf{r}_2) = \mu_2 \exp(-r'''/a),$$

$$M_{111}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \sum \lambda_i\lambda_j\lambda_k P_{ijk}(r', r'', r'''),$$

$$\mu_{111}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = M_{111}(r', r'', r''') - M_1[M_{11}(r') +$$

$$+ M_{11}(r'') + M_{11}(r''')] + 2M^3{}_1. \tag{2.5}$$

This expression gives

$$\frac{\partial}{\partial r'}\mu_{111}(r', r'', r''') =$$

$$= \frac{\partial}{\partial r'} M_{111}(r', r'', r''') - M_1 \frac{\partial}{\partial r'} M_{11}(r') \tag{2.6}$$

The derivatives on the right-hand side of (2.6) are defined from (1.11), (2.2), and (2.3), so

$$\frac{\partial}{\partial r'} M_{11}(r') = -\frac{1}{a}[M_{11}(r') - M_1^2]. \qquad (2.7)$$

$$\frac{\partial}{\partial r'} M_{111}(r', r'', r''') = -\frac{1}{a}\{M_{111}(r', r'', r''') -$$
$$- M_1[M_{11}(r'') + M_{11}(r''')] + M_1^3\} \qquad (2.8)$$

Substitution of (2.7) and (2.8) into (2.6) gives

$$\frac{\partial}{\partial r'} \mu_{111}(r', r'', r''') = -\frac{1}{a}\mu_{111}(r', r'', r'''). \qquad (2.9)$$

Analogous equations apply for differentiation with respect to other variables, so the central moment $\mu_{111}(r', r'', r''')$ takes the form

$$\mu_{111}(r', r'', r''') = \mu_{111}(0, 0, 0)\varphi_3(r', r'', r''') \qquad (2.10)$$

with

$$\mu_{111}(0, 0, 0) = \mu_3 \qquad (2.11)$$

$$\varphi_3(r', r'', r''') = \varphi(r')\varphi(r'')\varphi(r'''),$$
$$\varphi(r) = \exp(-r/a) \qquad (2.12)$$

From (1.12) and (2.10) we get the solution to (2.2) for the mixed probability $P_{ijk}(r_1, r_2, r_3)$:

$$\dot{P}_{ijk}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = c_ic_jc_k + c_i(c_j\delta_{jk} - c_jc_k)\varphi(r') +$$
$$+ c_j(c_i\delta_{ik} - c_ic_k)\varphi(r'') + c_k(c_i\delta_{ij} - c_ic_j)\varphi(r''') +$$
$$+ \Big[\sum_n c_n\delta_{in}\delta_{jn}\delta_{kn} - c_ic_j\delta_{jk} - c_jc_k\delta_{ik} - c_kc_i\delta_{ij} +$$
$$+ 2c_ic_jc_k\Big]\varphi(r')\varphi(r'')\varphi(r''') \qquad (2.13)$$

Expressions (2.10)–(2.12) also define the correlation functions for the elastic-modulus tensors; in fact, the second-order correlation function coincides with the central second moment, while $K_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ (third-order correlation function) is expressed as follows in terms of $\mu_{111}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ (central third moment):

$$K_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \mu_{111}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + 3M_1^3. \qquad (2.14)$$

The central moments of higher order can be derived similarly as

$$\mu_{\underbrace{11\ldots1}_{n}}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_n) = \mu_n \prod_{\alpha=1}^{0.5n(n-1)} \varphi(r^\alpha), \qquad (2.15)$$

in which $\mathbf{r}^\alpha \equiv \mathbf{r}_i - \mathbf{r}_j$, while the product is taken with respect to all possible pairs of differences $\mathbf{r}_i - \mathbf{r}_j$.

3. These results are used to calculate the effective bulk modulus for an untextured heterogeneous medium consisting of isotropic components. We envisage only the case in which the components have the same shear modulus.

A published technique [3–5] is used to renormalize the equation of equilibrium:

$$L_{il}u_l + f_i = 0, \qquad L_{il} \equiv \nabla_k\lambda_{iklm}\nabla_m. \qquad (3.1)$$

Summation with respect to subscripts occurring twice is assumed. The displacement vector u_l, the tensor λ_{iklm}, and hence operator L_{il}, are random functions of the point. We introduce the inverse operator M_{ln}

$$L_{il}M_{ln} = -\delta_{in}, \qquad (3.2)$$

and also the operators inverse to the regular components $\langle L_{il}\rangle$ and $\langle M_{ln}\rangle$

$$L_{il}^*\langle M_{ln}\rangle = -\delta_{in}, \qquad \langle L_{il}\rangle M_{ln}^* = -\delta_{in}, \qquad (3.3)$$

and thus find the renormalized equation of equilibrium for the regular part of the displacement vector:

$$L_{il}^*\langle u_l\rangle + f_i = 0. \qquad (3.4)$$

Here ⟨ ⟩ denotes averaging, and the averaging region is taken as fairly large relative to the spatial scale of the correlations, but small relative to the distance within which the regular parts of the functions change appreciably.

Matrix expressions give us the following for operator L^*:

$$L^* = \langle L\rangle - \Delta L, \qquad \Delta L = \langle L\rangle\frac{Q}{I+Q}, \qquad (3.5)$$

$$Q \equiv \langle M^*RM^*R\rangle + \langle M^*RM^*RM^*R\rangle + \ldots. \qquad (3.6)$$

Here I is the unit matrix, while R denotes the random component of L. Operator M^* is related as follows to the tensor Green's function for the quasi-homogeneous medium:

$$M_{ij}^*f_j = \int G_{ij}(\mathbf{r} - \mathbf{r}')f_j(\mathbf{r}')\,d\mathbf{r}' \equiv G_{ij} * f_j. \qquad (3.7)$$

We find from (3.5)–(3.7) that

$$\Delta L = \sum_0^\infty \Delta L^{(n)}, \qquad \Delta L^{(n)} = (-1)^n\langle L\rangle Q^{n+1}. \qquad (3.8)$$

The first two terms in the series of (3.8) are

$$\Delta L^{(0)} = -\langle RG * R\rangle - \langle RG * RG * R\rangle - \ldots, \qquad (3.9)$$

$$\Delta L^{(1)} = \langle RG * R\rangle G * \langle RG * R\rangle + 2\langle RG * R\rangle G *$$
$$* \langle RG * RG * R\rangle + \langle RG * RG * R\rangle G * \langle RG * RG * R\rangle + \ldots. \qquad (3.10)$$

We rearrange the terms in the double series of (3.8) to get

$$\Delta L = \sum_2^\infty \Delta K^{(n)} \qquad (3.11)$$

$$\Delta K^{(2)} = -\langle RG * R\rangle, \quad \Delta K^{(3)} = -\langle RG * RG * R\rangle$$
$$\Delta K^{(4)} = -\langle RG * RG * RG * R\rangle + \langle RG * R\rangle G * \langle RG * R\rangle$$
$$\Delta K^{(5)} = -\langle RG * RG * RG * RG * R\rangle +$$
$$+ 2\langle RG * R\rangle G * \langle RG * RG * R\rangle \qquad (3.12)$$

Here $\Delta K^{(n)}$ combines all terms of (3.8) that contain the random operator R n times.

We calculate $\Delta K^{(2)}$ and the second term of $\Delta K^{(4)}$; the other terms in $\Delta K^{(n)}$ can be derived similarly.

We substitute the explicit values of the R into $\Delta K^{(2)}$ to get

$$-\langle R_{ik}G_{kn} * R_{nl}\rangle\langle u_l\rangle =$$
$$= -\nabla_j\langle\delta\lambda_{ijkm}\nabla_mG_{kn} * \nabla_p\delta\lambda_{nplq}\nabla_q\rangle\langle u_l\rangle =$$
$$= \nabla_jA_{nplq}^{ijkm}\int\varphi(\mathbf{r} - \mathbf{r}')\;G_{kn,\,mp}(\mathbf{r} - \mathbf{r}')\langle u_{l,\,q}(\mathbf{r}')\rangle\,d\mathbf{r}' \qquad (3.13)$$

Here $\delta\lambda_{ijkm} = \lambda_{ijkm} - \langle\lambda_{ijkm}\rangle$ is the fluctuating part of the elastic-modulus tensor and $A_{nplq}^{ijkm}\,\varphi(\mathbf{r})$ is the pair correlation function, with $\varphi(\mathbf{r})$ defined by (2.5) and (2.12).

As only the bulk modulus K fluctuates here, the following is the autocorrelation tensor for the elastic moduli:

$$A_{nplq}^{ijkm} = D_K^{(2)}\delta_{ij}\delta_{km}\delta_{np}\delta_{lq}, \qquad D_K^{(2)} \equiv \langle(\delta K)^2\rangle. \qquad (3.14)$$

We thus need to know the convolution $G_{kn,\,kn}$ in order to calculate (3.13), and this is found via a chain of equations:

$$G_{ln,\,ln}=\frac{1}{8\pi\mu}\left[\frac{2}{r}\delta_{kn}-\frac{3\langle K\rangle+\mu}{3\langle K\rangle+4\mu}r_{,\,kn}\right]_{,\,ln}=$$

$$=\frac{1}{4\pi(\langle K\rangle+{}^4/_3\mu)}\Delta\frac{1}{r}=-\frac{1}{\langle K\rangle+{}^4/_3\mu}\delta(\mathbf{r}) \qquad (3.15)$$

We substitute (3.14) and (3.15) into (3.13) to get

$$\Delta K_{il}^{(2)}=\nabla_i\nabla_l D_K^{(2)}(\langle K\rangle+{}^4/_3\mu)^{-1}. \qquad (3.16)$$

A similar calculation gives as follows for the first terms of the $\Delta K^{(n)}$:

$$-\langle R_{ik}G_{km}*R_{mp}G_{pq}*\ldots R_{sl}\rangle=(-1)^n\nabla_i\nabla_l,$$

$$D_K^{(n)}(\langle K\rangle+{}^4/_3\mu)^{1-n}. \qquad (3.17)$$

Here n equals the number of cofactors R_{ik} on the left, while the central moments of order n for K are defined by

$$D_K^{(1)}=0,\quad D_K^{(2)}=c_1c_2(K_1-K_2)^2,\ldots,$$

$$D_K^{(n)}=c_1c_2[c_2^{\,n-1}+(-1)^n c_1^{\,n-1}]\,(K_1-K_2)^n. \qquad (3.18)$$

Consider now the second term in ΔK_4. We use (3.13) to get

$$\langle R_{ik}G_{kn}*R_{nr}\rangle G_{rs}*\langle R_{su}G_{uv}*R_{vl}\rangle\langle u_l\rangle=$$

$$=\nabla_j A_{nprq}^{ijkm}A_{valb}^{stuw}\iiint\varphi(\mathbf{r}-\mathbf{r}')\,\varphi(\mathbf{r}''-\mathbf{r}''')\,G_{ln,\,mp}(\mathbf{r}-\mathbf{r}'),$$

$$G_{rs,\,qt}(\mathbf{r}'-\mathbf{r}'')\,G_{uv,\,wa}(\mathbf{r}''-\mathbf{r}''')\,\langle u_{l,\,b}(\mathbf{r}''')\rangle\,d\mathbf{r}'\,d\mathbf{r}''\,d\mathbf{r}'''. \qquad (3.19)$$

We use (3.14) and (3.15), and also integrate, to get

$$\langle R_{ik}G_{kn}*R_{nr}\rangle G_{rs}*\langle R_{su}G_{uv}*R_{vl}\rangle=$$

$$=-\nabla_i\nabla_l(D_K^{(2)})^2(\langle K\rangle+{}^4/_3\mu)^{-3}. \qquad (3.20)$$

The other terms of the $\Delta K^{(n)}$ are calculated similarly; this together with (3.16), (3.18), and (3.20) gives

$$\Delta K_{il}^{(n)}=\nabla_i\nabla_l\varkappa^{(n)}, \qquad (3.21)$$

$$\varkappa^{(2)}=\frac{D_K^{(2)}}{\langle K\rangle+{}^4/_3\mu},\qquad \varkappa^{(n)}=\varkappa^{(2)}\xi^{n-2},$$

$$\xi\equiv\frac{(c_1-c_2)(K_1-K_2)}{\langle K\rangle+{}^4/_3\mu}. \qquad (3.22)$$

We see from (3.21) that summation of the operator series of (3.11) amounts to summation of the numerical series

$$\Delta L_{il}=\nabla_i\nabla_l\sum_2^\infty\varkappa^{(n)}=\nabla_i\nabla_l\frac{\varkappa^{(2)}}{1-\xi}. \qquad (3.23)$$

We put the effective operator L_{il}^* as

$$L_{il}{}^*=(K^*+{}^1/_3\mu)\nabla_i\nabla_l+\mu\delta_{il}\nabla^2 \qquad (3.24)$$

and use (3.5) and (3.23) to get K^* (the effective bulk modulus):

$$K^*=\langle K\rangle-\frac{D_K^{(2)}}{c_1K_2+c_2K_1+{}^4/_3\mu}. \qquad (3.25)$$

These results are restricted by the condition for (3.23) to converge:

$$|\xi|<1. \qquad (3.26)$$

This condition is met for any component concentrations and arbitrary values of the elastic moduli. In the limit $c_2 \to 0$ we have $|\xi| = 1$, but then $\varkappa^{(2)} = 0$, and hence $\varkappa^{(n)} = 0$.

4. This scheme is based on calculating the correlation corrections to $\langle\lambda_{iklm}\rangle$. Another approach to allowance for the correlations consists in finding the correlation corrections to the tensor for the s_{iklm}. The basis of the calculation is

$$L_{iklm}\sigma_{lm}+\eta_{ik}=0,\qquad L_{iklm}\equiv\varepsilon_{ipq}\varepsilon_{krs}\nabla_p\nabla_r s_{qslm}. \qquad (4.1)$$

Here σ_{lm} is the stress tensor, η_{ik} is the incompatibility tensor, and ε_{ipq} is unit antisymmetric tensor.

In [6] L_{iklm} was renormalized in the second approximation. The above method for renormalizing L_{il} gives

$$\overset{*}{s}_{iklm}=\frac{1}{9K^*}\delta_{ik}\delta_{lm}+ \qquad (4.2)$$

$$+\frac{1}{4\mu}\left(\delta_{il}\delta_{km}+\delta_{im}\delta_{kl}-\frac{2}{3}\delta_{ik}\delta_{lm}\right),$$

$$\frac{1}{K^*}=\left\langle\frac{1}{K}\right\rangle-\frac{D_{1/K}^{(2)}}{c_1/K_2+c_2/K_1+0.75/\mu}. \qquad (4.3)$$

The following convergence condition was used instead of (3.26) in deducing (4.3):

$$|\eta|<1,\qquad \eta\equiv\frac{(c_1-c_2)(1/K_1-1/K_2)}{\langle 1/K\rangle+0.75/\mu}. \qquad (4.4)$$

Since K^* has been calculated with allowance for all correlations, the K^* given by (3.25) and (4.3) should coincide, which is readily seen if we note that

$$D_K^{(2)}=D_{1/K}^{(2)}K_1^2K_2^2.$$

This K^* defines the exact relation between the averaged bulk stresses and strains:

$$\langle\sigma_{ii}\rangle=3K^*\langle\varepsilon_{kk}\rangle.$$

The values $\langle K\rangle$ and $\langle 1/K\rangle^{-1}$ define the upper and lower bounds to the effective modulus. Formulas (3.25) and (4.3) show that this requirement is met for any concentrations and elastic moduli, with K^* coincident with the upper bound $\langle K\rangle$, if $\mu\to\infty$ and with the lower bound $\langle 1/K\rangle^{-1}$ if $\mu\to 0$.

The effective modulus coincides with the arithmetic mean ${}^1/_2(\langle K\rangle+\langle 1/K\rangle^{-1})$ if $4\mu/3 = c_1K_2 + c_2K_1$.

Hill [7] used energy considerations for the case $\mu_1 = \mu_2$ to calculate the true bulk modulus as

$$K=K_R\left[1+\frac{4}{3}\frac{\mu K_V}{K_1K_2}\right]\left[1+\frac{4}{3}\frac{\mu K_R}{K_1K_2}\right]^{-1},$$

$$K_V\equiv\langle K\rangle,\qquad K_R\equiv\langle 1/K\rangle^{-1}. \qquad (4.6)$$

It is readily seen that (4.6) coincides with K^*.

The Hashin-Shtrikman variational method [8] provides more closely spaced bounds than does direct Voigt-Reuss averaging; only for $\mu_1 = \mu_2$ does the method give an exact value for the bulk modulus that coincides with (4.6) and (3.25). On the other hand, the use of random functions allows one in principle to calculate exact values for the effective moduli K and μ if the fluctuations allow of expansion as in (3.8).

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REFERENCES

1. M. A. Krivoglaz and A. S. Cherevko, "The elastic moduli of mixtures," Fizika metallov i metallovedenie, 8, no. 2, 1959.

2. I. M. Lifshits and L. N. Rozentsveig, "Theory of the elastic properties of polycrystals," ZhETF, 16, no. 11, 1946.

3. V. I. Tatarskii, Wave Propagation in a Turbulent Atmosphere [in Russian], Izd-vo Nauka, 1967.

4. V. M. Finkel'berg, "Dielectric constants of mixtures," ZhTF, 34, no. 3, 1964.

5. B. M. Darinskii and T. D. Shermergor, "Temperature relaxation in cubic polycrystals," Fizika metallov i metallovedenie, 18, no. 5, 1964.

6. B. M. Darinskii and T. D. Shermergor, "Elastic moduli of cubic polycrystals," PMTF [Journal of Applied Mechanics and Technical Physics], no. 4,

1965.

7. R. Hill, "Elastic properties of reinforced solids, some theoretical principles," J. Mech. and Phys. Solids, vol. 11, no. 5, 1963.

8. Z. Hashin, and S. Shtrikman, "A variational approach of the theory of the elastic behaviour of multiphase materials," J. Mech. and Phys. Solids, vol. 11, no. 2, 1963.

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